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Existence and uniqueness of weak solutions for a non-uniformly parabolic equation ☆

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Abstract

In this paper we develop a unifying method to prove the existence and uniqueness of weak solutions for the initial–boundary value problem of a non-uniformly parabolic equation. Some well-known parabolic equations are the special cases of this equation.

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1. Introduction

Suppose that Ω is a bounded domain of $\mathbb{R}^N$ $(N \geqslant 2)$ with Lipschitz boundary $\partial\Omega$, and T is a positive number. Denote $\Omega_T = \Omega \times (0, T]$. In this paper we study the well-posedness for the initial–boundary value problem of the following non-uniformly parabolic equation

$$u_t - \operatorname{div}\big(D_\xi \Phi(\nabla u)\big) = 0 \quad \text{in } \Omega_T, \tag{1.1}$$

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where $\Phi(\xi):\mathbb{R}^N\to\mathbb{R}_+$ is a C^1 nonnegative convex function, $D_\xi\Phi(\xi):\mathbb{R}^N\to\mathbb{R}^N$ represents the gradient of $\Phi(\xi)$ with respect to ξ and ∇u represents the gradient with respect to the spatial variables x. Without loss of generality we may assume that $\Phi(0)=0$.

Our main assumptions are that $\Phi(\xi)$ satisfies the super-linear condition (1-coercive condition, see [16, Chapter E])

$$\lim_{|\xi|\to+\infty}\frac{\Phi(\xi)}{|\xi|}=+\infty, \tag{1.2}$$

and the symmetric condition: there exists a positive number $C>0$ such that

$$\Phi(-\xi)\leqslant C\Phi(\xi),\quad \xi\in\mathbb{R}^N. \tag{1.3}$$

The heat equation is the simplest form of problem (1.1). The similar types of Eq. (1.1) have been extensively investigated. We refer readers to [20,18,9,17] and the reference therein. There are numerous examples of $\Phi(\xi)$ satisfying structure assumptions (1.2) and (1.3). The well known are listed as follows.

Example 1.

$$\Phi(\xi)=\frac{1}{p}|\xi|^p,\quad p>1.$$

In this case, Eq. (1.1) is

$$u_t-\Delta_p u=0, \tag{1.4}$$

which is the parabolic counterpart of the p-Laplacian. Here $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator. Eq. (1.4) is a variant of Navier–Stokes equation to describe the motion of non-Newtonian fluids, whose velocity gradient depends nonlinearly on the stress tensor. The p-Laplacian type equations have been thoroughly studied these decades and there are many applications in fluid mechanics, glaciology, and rheology, etc. (See [9] and [20, Chapter 2].)

Example 2.

$$\Phi(\xi)=\frac{1}{p_1}|\xi_1|^{p_1}+\frac{1}{p_2}|\xi_2|^{p_2}+\cdots+\frac{1}{p_N}|\xi_N|^{p_N},\quad p_i>1,\ i=1,2,\ldots,N,$$

where $\xi=(\xi_1,\xi_2,\ldots,\xi_N)$. In this case, Eq. (1.1) is the parabolic counterpart of the anisotropic p-Laplacian. (See [20, Chapter 2].)

Example 3.

$$\Phi(\xi)=|\xi|\log\bigl(1+|\xi|\bigr).$$

This special case has been investigated thoroughly in [29] as a model that developed Perona and Malik's idea in [22]. $L\log L$ type functions arise naturally in the research of the entropies of systems. (See [10] and [6, Chapter 4].)

Example 4.

$$\Phi(\xi) = |\xi| L_k\big(|\xi|\big),$$

where $L_i(s) = \log(1 + L_{i-1}(s))$ $(i = 1, 2, \dots, k)$ and $L_0(s) = \log(1 + s)$, for $s \geqslant 0$. The corresponding elliptic problems are introduced in Prandtl–Eyring fluids and plastic materials with logarithmic hardening law. (See [15].)

Example 5.

$$\Phi(\xi) = e^{\frac{|\xi|^2}{2}} - 1.$$

The corresponding elliptic case, which originated from the exponential harmonic mappings has been studied in [21,12,17], especially the regularity theory. Naito [21] proved existence, uniqueness and C^α regularity of the minimizer. Duc and Eells [12], Lieberman [17] respectively proved the C^∞ or $C^{1,\alpha}$ regularity of the minimizer. Lieberman [19] proved the interior $C^{1,\alpha}$-estimate for the parabolic counterpart.

Siepe [27] has proved Lipschitz regularity of the minimizers of functional $\int_\Omega \Psi(\nabla u)\,dx$, under the main assumption that implies the Δ_2 condition on Ψ: there exists a positive number $K > 2$ such that

$$\Phi(2\xi) \leqslant K\Phi(\xi).$$

Dong [11] used the Galerkin method to study an elliptic system with more general structure, which is similar to the elliptic counterpart of (1.1). The existence of weak solutions in Orlicz–Sobolev space has been obtained without assuming the Δ_2 condition. However, the uniqueness result is unknown.

When those parabolic problems or the corresponding variational problems were studied, growth conditions such as polynomial growth or exponential growth were usually assumed for function $\Phi(\xi)$. (See [1,5,17].) Generally speaking, finding solutions for such parabolic problems or deriving the Euler–Lagrange equations for minimizers of variational problems is not a trivial fact when function $\Phi(\xi)$ does not satisfy the Δ_2 condition.

There are some well-known models in image processing which can be reduced to Eq. (1.1). Perona and Malik [22] proposed the nonlinear diffusion equation

$$u_t - \operatorname{div}\Big(c\big(|\nabla u|^2\big)\nabla u\Big) = 0 \tag{1.5}$$

to denoise images in image processing, where $c(s) = (1 + s/K)^{-1}$, or $c(s) = \exp(-s/K)$, and K is a given threshold. To guarantee the parabolicity of (1.5), $c(s)$ is usually chosen as a decreasing positive function satisfying $c(s) \approx 1/\sqrt{s}$ as $s \to +\infty$, and $c(s) + 2sc'(s) > 0$. A canonical example is $c(s) = (s + 1)^{-1/2}$. When we choose that

$$\Phi(\xi) = \frac{1}{2}\int_0^{|\xi|^2} c(s)\,ds,$$

Eq. (1.1) reads as Eq. (1.5). However, functions $\Phi(\xi)$ as above do not satisfy (1.2). (See [3,4,26,28] for further discussion about the related topics.)

In this paper we assume the solution of Eq. (1.1) satisfies the following initial–boundary conditions:

$$u|_{\Sigma} = 0 \tag{1.6}$$

in the trace sense, where $\Sigma = \partial\Omega \times (0, T]$ and

$$u|_{t=0} = u_0 \in L^2(\Omega). \tag{1.7}$$

Now we define weak solutions of Eq. (1.1) with initial–boundary conditions (1.6) and (1.7). Our weak solutions are more restrictive than the usual weak solutions.

Definition 1.1. A function $u : \bar{\Omega} \times [0, T] \to \mathbb{R}$ is a weak solution of Eq. (1.1) with initial–boundary conditions (1.6) and (1.7) if the following conditions are satisfied:

(i) $u \in C([0, T]; L^2(\Omega)) \cap L(0, T, W_0^{1,1}(\Omega))$ with

$$\int_0^T \int_\Omega D_\xi \Phi(\nabla u) \cdot \nabla u \, dx \, dt < +\infty;$$

(ii) For every $\varphi \in C^1(\bar{\Omega}_T)$ with $\varphi(\cdot, T) = 0$ and $\varphi(\cdot, t)|_{\partial\Omega} = 0$, we have

$$-\int_\Omega u_0(x)\varphi(x, 0) \, dx + \int_0^T \int_\Omega \left[-u\varphi_t + D_\xi \Phi(\nabla u) \cdot \nabla\varphi\right] dx \, dt = 0. \tag{1.8}$$

Next, we state our main theorem.

Theorem 1.2. *Under structure assumptions* (1.2), (1.3)*, there exists a unique weak solution for Eq.* (1.1) *with initial–boundary conditions* (1.6) *and* (1.7).

Remark 1.3. Let u be a weak solution in Definition 1.1. By using the approximation technique (see [7, Chapter 3] or [9, Chapter 2]) we have, for every $\varphi \in C^1(\bar{\Omega}_T)$ with $\varphi(\cdot, t)|_{\partial\Omega} = 0$, each $t \in [0, T]$,

$$\int_\Omega u\varphi \, dx \Big|_0^t + \int_0^t \int_\Omega \left[-u\varphi_t + D_\xi \Phi(\nabla u) \cdot \nabla\varphi\right] dx \, d\tau = 0. \tag{1.9}$$

The condition (i) in Definition 1.1 is crucial in two ways. It guarantees the uniqueness of weak solutions. And it ensures us to obtain an energy type equality by choosing solution u as a test function in (1.9).

Corollary 1.4. *Let u be a weak solution of problem* (1.1). *Then we have, for every* $t \in (0, T]$,

$$\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega D_\xi \Phi(\nabla u) \cdot \nabla u \, dx \, d\tau = \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}. \tag{1.10}$$

Remark 1.5. The same arguments may be applied to obtain the well-posedness for Eq. (1.1) with the same initial condition and Neumann boundary condition, and to deal with Eq. (1.1) with lower order terms satisfying suitable growth conditions and integrabilities.

Inspired by the ideas in [13,20,23,30], we develop a unifying method to prove the existence and uniqueness of weak solutions for nonuniformly parabolic equation (1.1). The novelties in this paper are mainly two parts. First, we do not assume polynomial or exponential growth for function Φ as in [1,5,17]. Second, we provide an approximation argument to study this kind of problems by finding a weak limit for approximation solution sequence with bounded L^1-norm under certain conditions and then proving this limit is a weak solution.

This paper is organized as follows. In Section 2, we will list some useful lemmas. In Section 3, we prove the main results. To prove the uniqueness of weak solutions of problem (1.1), we choose suitable test functions and then take the limit to get the conclusion. To prove the existence result, we first combine the difference and variation techniques to find a unique minimizer in a special function class for a functional and then prove the minimizer satisfies the corresponding Euler–Lagrange equation. Then we construct an approximation solution sequence for problem (1.1) and establish *a priori* estimates. Next, we draw a subsequence to obtain a limit function, and then prove this function is a weak solution. Next we prove the energy type estimate (1.10) by an approximation argument.

In the following sections C will represent a generic constant that may change from line to line even if in the same inequality.

2. Inequalities and lemmas

Let $\Phi(\xi)$ be a nonnegative convex function. We define the polar function of $\Phi(\xi)$ as

$$\Psi(\eta) = \sup_{\xi \in \mathbb{R}^N} \{\eta \cdot \xi - \Phi(\xi)\}, \tag{2.1}$$

which is also known as the Legendre transform of $\Phi(\xi)$. It is obvious that $\Psi(\eta)$ is a convex function. In the following we list several lemmas.

Lemma 2.1. *Suppose* $\Phi : \mathbb{R}^N \to \mathbb{R}$ *is a convex* C^1 *function with* $\Phi(0) = 0$. *Then we have, for all* $\xi, \zeta \in \mathbb{R}^N$,

$$\Phi(\xi) \leqslant \xi \cdot D\Phi(\xi), \tag{2.2}$$

$$\big(D\Phi(\zeta) - D\Phi(\xi)\big) \cdot (\zeta - \xi) \geqslant 0. \tag{2.3}$$

Lemma 2.2. *Suppose* $\Phi(\xi)$ *is a nonnegative convex* C^1 *function and* $\Psi(\eta)$ *is its polar function. Then we have, for all* $\xi, \eta, \zeta \in \mathbb{R}^N$,

$$\xi \cdot \eta \leqslant \Phi(\xi) + \Psi(\eta), \tag{2.4}$$

$$\Psi\big(D\Phi(\zeta)\big) + \Phi(\zeta) = D\Phi(\zeta) \cdot \zeta. \tag{2.5}$$

Proof. The first one follows directly from (2.1). As $\Phi(\xi)$ is a convex C^1 function, we have, for every $\xi \in \mathbb{R}^N$,

$$\Phi(\xi) \geqslant \Phi(\zeta) + D\Phi(\zeta) \cdot (\xi - \zeta),$$

which implies

$$D\Phi(\zeta)\zeta - \Phi(\zeta) \geqslant D\Phi(\zeta) \cdot \xi - \Phi(\xi).$$

Recalling

$$\Psi\big(D\Phi(\zeta)\big) = \sup_{\xi \in \mathbb{R}^n} \big\{D\Phi(\zeta) \cdot \xi - \Phi(\xi)\big\}$$

and inequality (2.2), we conclude that

$$\Psi\big(D\Phi(\zeta)\big) = D\Phi(\zeta) \cdot \zeta - \Phi(\zeta).$$

This completes the proof. □

Lemma 2.3. *Suppose $\Phi : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative convex function with $\Phi(0) = 0$, which satisfies* (1.2)*. Then $\Psi(\eta)$ in* (2.1) *is a well defined, nonnegative function in $\mathbb{R}^N$.* (*See* [14, *Chapter* 3].)

Lemma 2.4. *If a convex function $\Phi : \mathbb{R}^N \to \mathbb{R}$ satisfies condition* (1.2)*, then its polar function Ψ also satisfies condition* (1.2)*.* (*See* [14, *Chapter* 3].)

Lemma 2.5. *Let $D \subset \mathbb{R}^N$ be measurable with finite Lebesgue measure and $f_k \in L^1(D)$ and $g_k \in L^1(D)$ $(k = 1, 2, \ldots)$, and*

$$\big|f_k(x)\big| \leqslant g_k(x), \quad \text{a.e. } x \in D,\ k = 1, 2, \ldots.$$

If

$$\lim_{k\to\infty} f_k(x) = f(x), \qquad \lim_{k\to\infty} g_k(x) = g(x), \quad \text{a.e. } x \in D,$$

and

$$\lim_{k\to\infty} \int_D g_k(x)\,dx = \int_D g(x)\,dx < +\infty,$$

then

$$\lim_{k\to\infty}\int_D f_k(x)\,dx=\int_D f(x)\,dx.$$

(*See* [24, *Chapter* 4].)

Lemma 2.6. *Suppose $\Phi(\xi)$ is a nonnegative convex function satisfying* (1.2). *Let $D\subset\mathbb{R}^N$ be measurable with finite Lebesgue measure and a sequence $\{f_k\}\subset L^1(D;\mathbb{R}^N)$ satisfy that*

$$\int_D \Phi(f_k)\,dx\leqslant C, \tag{2.6}$$

where C is a positive constant. Then there exist a subsequence $\{f_{k_j}\}\subset\{f_k\}$ and a function $f\in L^1(D;\mathbb{R}^N)$ such that

$$f_{k_j}\rightharpoonup f\quad \textit{weakly in } L^1\big(D;\mathbb{R}^N\big)\textit{ as } j\to\infty \tag{2.7}$$

and

$$\int_D \Phi(f)\,dx\leqslant\liminf_{j\to\infty}\int_D \Phi(f_{k_j})\,dx\leqslant C. \tag{2.8}$$

(*See* [8, *Chapter* 3] *and* [25].)

3. Existence and uniqueness

Before proving Theorem 1.2, we first prove the existence and uniqueness of weak solutions of the following elliptic problems

$$\begin{cases}\dfrac{u-u_0}{h}-\operatorname{div}\big(D_\xi\Phi(\nabla u)\big)=0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{3.1}$$

where $h>0$ and $u_0\in L^2(\Omega)$.

Definition 3.1. A function $u\in L^2(\Omega)\cap W_0^{1,1}(\Omega)$ with $D_\xi\Phi(\nabla u)\cdot\nabla u\in L^1(\Omega)$ is called a weak solution of problem (3.1) if for every $\varphi\in C_0^1(\Omega)$, we have

$$\int_\Omega \frac{u-u_0}{h}\varphi\,dx+\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla\varphi\,dx=0. \tag{3.2}$$

Remark 3.2. The requirement $D_\xi\Phi(\nabla u)\cdot\nabla u\in L^1(\Omega)$ makes it possible to find an energy type estimate and prove the uniqueness of solutions.

Theorem 3.3. *There exists a unique weak solution for problem* (3.1).

Before we prove this theorem, let us mention Proposition 2.1 in [21], where the same result has been proved when $\Phi(\xi) = e^{|\xi|^2}$ like Example 5.

Proof. We consider the variational problem $\min\{J(v) \mid v \in V\}$, where $V = \{v \in L^2(\Omega) \cap W_0^{1,1}(\Omega) \mid \Phi(\nabla v) \in L^1(\Omega)\}$, and

$$J(v) = \frac{1}{2h}\int_\Omega (v - u_0)^2\,dx + \int_\Omega \Phi(\nabla v)\,dx.$$

We will establish that $J(v)$ has a minimizer $u_1(x)$ in V and then prove that the minimizer satisfies the Euler–Lagrange equation of functional J weakly.

As

$$0 \leqslant \inf_{v\in V} J(v) \leqslant J(0) = \frac{1}{2h}\int_\Omega u_0^2\,dx,$$

we can find a minimizing sequence $\{v_m\} \subset V$ such that $\lim_{m\to\infty} J(v_m) = \inf_{v\in V} J(v)$. Thus we have

$$\int_\Omega v_m^2\,dx + \int_\Omega \Phi(\nabla v_m)\,dx \leqslant C.$$

By using Lemma 2.6 we may find a subsequence $\{v_{m_i}\}$ of $\{v_m\}$ and a function $u_1 \in L^2(\Omega) \cap W_0^{1,1}(\Omega)$ such that

$$\begin{aligned} v_{m_i} &\rightharpoonup u_1 \quad \text{weakly in } L^2(\Omega),\\ \nabla v_{m_i} &\rightharpoonup \nabla u_1 \quad \text{weakly in } L^1(\Omega). \end{aligned}$$

Therefore, we obtain

$$J(u_1) \leqslant \liminf_{i\to\infty} J(v_{m_i}) = \inf_{v\in V} J(v).$$

This implies $u_1 \in V$ is a minimizer of the functional $J(u)$ in V.

Furthermore, we have $J(u_1) \leqslant J(\lambda u_1)$, $\lambda \in (0,1)$. Recalling (2.3), we know

$$\Phi(\nabla u_1) - \Phi(\lambda\nabla u_1) \geqslant (1-\lambda) D_\xi \Phi(\lambda\nabla u_1)\cdot\nabla u_1,$$

then

$$\frac{1}{2}\left(1-\lambda^2\right)\int_\Omega u_1^2\,dx + h(1-\lambda)\int_\Omega D_\xi\Phi(\lambda\nabla u_1)\cdot\nabla u_1\,dx \leqslant (1-\lambda)\int_\Omega u_1 u_0\,dx.$$

Dividing the above inequality by $1-\lambda$, and passing to limits as $\lambda \to 1$, we have

$$\int_\Omega u_1^2\,dx + h\liminf_{\lambda\to 1}\int_\Omega D_\xi\Phi(\lambda\nabla u_1)\cdot\nabla u_1\,dx \leqslant \int_\Omega u_1u_0\,dx.$$

Since $D_\xi\Phi(\lambda\nabla u_1)\cdot\nabla u_1 \geqslant 0$, by Fatou's Lemma we conclude

$$\int_\Omega u_1^2\,dx + h\int_\Omega D_\xi\Phi(\nabla u_1)\cdot\nabla u_1\,dx \leqslant \int_\Omega u_1u_0\,dx. \tag{3.3}$$

It follows from (2.5) that $D_\xi\Phi(\nabla u_1)\cdot\nabla u_1 \in L^1(\Omega)$ and $\Psi(D_\xi\Phi(\nabla u_1)) \in L^1(\Omega)$.

For a fixed $\varphi(x) \in C_0^1(\Omega)$, we know that $J(u_1) \leqslant J(\lambda u_1 + (1-\lambda)\varphi)$, $\lambda \in (0,1)$.

Denote $\xi_\lambda = \lambda\nabla u_1 + (1-\lambda)\nabla\varphi$. In light of (2.3), we find

$$\Phi(\nabla u_1) - \Phi(\xi_\lambda) \geqslant (1-\lambda)D_\xi\Phi(\xi_\lambda)\cdot(\nabla u_1 - \nabla\varphi),$$

and deduce as above to have

$$\begin{aligned}\int_\Omega D_\xi\Phi(\xi_\lambda)\cdot(\nabla u_1-\nabla\varphi)\,dx \leqslant \frac{1}{2h}\int_\Omega \big[&-(1+\lambda)(u_1-u_0)^2 + 2\lambda(u_1-u_0)(\varphi-u_0)\\ &+(1-\lambda)(\varphi-u_0)^2\big]\,dx.\end{aligned} \tag{3.4}$$

Consider

$$g(\lambda) = \Phi(\xi_\lambda) = \Phi\big(\lambda\nabla u_1 + (1-\lambda)\nabla\varphi\big).$$

It is obvious that g is a convex function in $\mathbb{R}$. Then by the monotonicity of a convex function's derivative, we know

$$g'(0) \leqslant g'(\lambda) \leqslant g'(1), \quad \lambda \in (0,1),$$

which implies that

$$D_\xi\Phi(\nabla\varphi)\cdot(\nabla u_1-\nabla\varphi) \leqslant D_\xi\Phi(\xi_\lambda)\cdot(\nabla u_1-\nabla\varphi) \leqslant D_\xi\Phi(\nabla u_1)(\nabla u_1-\nabla\varphi). \tag{3.5}$$

Recalling (2.4), (1.3) and (2.5), we have

$$\begin{aligned}\big|D_\xi\Phi(\nabla u_1)\cdot\nabla\varphi\big| &\leqslant \Psi\big(D_\xi\Phi(\nabla u_1)\big) + \Phi(\nabla\varphi) + \Phi(-\nabla\varphi)\\ &\leqslant \Psi\big(D_\xi\Phi(\nabla u_1)\big) + (C+1)\Phi(\nabla\varphi).\end{aligned} \tag{3.6}$$

As $\Psi(D_\xi\Phi(\nabla u_1)) \in L^1(\Omega)$ and $\varphi \in C_0^1(\Omega)$, it is easy to know $D_\xi\Phi(\nabla\varphi)\cdot(\nabla u_1-\nabla\varphi) \in L^1(\Omega)$ and $D_\xi\Phi(\nabla u_1)(\nabla u_1-\nabla\varphi) \in L^1(\Omega)$. By Lebesgue dominated convergence theorem, we have

$$\int_\Omega \lim_{\lambda\to 1} D_\xi \Phi(\xi_\lambda)\cdot(\nabla u_1-\nabla\varphi)\,dx=\lim_{\lambda\to 1}\int_\Omega D_\xi \Phi(\xi_\lambda)\cdot(\nabla u_1-\nabla\varphi)\,dx.$$

Recalling (3.4), we obtain

$$\int_\Omega D_\xi \Phi(\nabla u_1)\cdot(\nabla u_1-\nabla\varphi)\,dx\leqslant \frac{1}{h}\int_\Omega (u_1-u_0)(\varphi-u_1)\,dx.$$

Denote

$$A_0=\int_\Omega \frac{u_1-u_0}{h}u_1\,dx+\int_\Omega D_\xi \Phi(\nabla u_1)\cdot\nabla u_1\,dx.$$

Then we conclude that, for every $\varphi(x)\in C_0^1(\Omega)$,

$$\int_\Omega \frac{u_1-u_0}{h}\varphi\,dx+\int_\Omega D_\xi \Phi(\nabla u_1)\cdot\nabla\varphi\,dx\geqslant A_0. \tag{3.7}$$

By a scaling argument, it follows that

$$\int_\Omega \frac{u_1-u_0}{h}\varphi\,dx+\int_\Omega D_\xi \Phi(\nabla u_1)\cdot\nabla\varphi\,dx=0. \tag{3.8}$$

Therefore, $u_1(x)$ is a weak solution of problem (3.1). By an approximation argument, we conclude that

$$\int_\Omega \frac{u_1-u_0}{h}u_1\,dx+\int_\Omega D_\xi \Phi(\nabla u_1)\cdot\nabla u_1\,dx=0. \tag{3.9}$$

Suppose that there exists another weak solution u^1 of problem (3.1). Then, for every $\varphi\in C_0^1(\Omega)$, we have

$$\int_\Omega \frac{u^1-u_0}{h}\varphi\,dx+\int_\Omega D_\xi \Phi\big(\nabla u^1\big)\cdot\nabla\varphi\,dx=0,$$

which follows that

$$\int_\Omega \frac{u^1-u_1}{h}\varphi\,dx+\int_\Omega \big[D_\xi \Phi\big(\nabla u^1\big)-D_\xi \Phi(\nabla u_1)\big]\cdot\nabla\varphi\,dx=0, \tag{3.10}$$

for every $\varphi\in C_0^1(\Omega)$. Recalling (2.4) and (2.5) we observe that

$$\begin{aligned}\left|D_\xi \Phi\left(\nabla u^1\right)\cdot \nabla u_1\right| &\leqslant \Phi(\nabla u_1)+\Phi(-\nabla u_1)+\Psi\left(D_\xi \Phi\left(\nabla u^1\right)\right)\\ &\leqslant \Phi(\nabla u_1)+\Phi(-\nabla u_1)+D_\xi \Phi\left(\nabla u^1\right)\cdot \nabla u^1 \in L^1(\Omega).\end{aligned}$$

Making use of the approximation argument, we conclude that $w=u^1-u_1$ can be a test function in (3.10). Therefore,

$$\int_\Omega \frac{(u^1-u_1)^2}{h}\,dx+\int_\Omega \left[D_\xi \Phi\left(\nabla u^1\right)-D_\xi \Phi(\nabla u_1)\right]\cdot\left(\nabla u^1-\nabla u_1\right)dx=0.$$

Using inequality (2.3), we have

$$\int_\Omega \left(u^1-u_1\right)^2 dx=0,$$

which implies $u^1=u_1$ a.e. in Ω. Thus we complete the proof of the theorem. □

Proof of Theorem 1.2. First we prove the uniqueness of weak solutions. Suppose there exist two weak solutions u and v of problem (1.1). Then $w=u-v$ satisfies the following problem

$$\begin{cases} w_t-\operatorname{div}\left[D_\xi \Phi(\nabla u)-D_\xi \Phi(\nabla v)\right]=0 & \text{in } \Omega_T,\\ w(x,t)=0, & \text{on } \Sigma,\\ w(x,0)=0, & \text{in } \Omega.\end{cases}$$

Using the approximation argument in Corollary 1.4, we choose

$$\omega^k_{\varepsilon,h}(x,t)=\frac{1}{2h}\int_{t-h}^{t+h}\omega^k_\varepsilon(x,\tau)\,d\tau,\qquad \omega^k=\omega\chi_{\{|\omega|\leqslant k\}}-k\chi_{\{\omega<-k\}}+k\chi_{\{\omega>k\}}$$

as a test function in the above initial–boundary value problem to have

$$\int_\Omega \left[w\omega^k_{\varepsilon,h}\right](x,t)\,dx-\int_0^t\int_\Omega w\left[\omega^k_{\varepsilon,h}\right]_t dt+\int_0^t\int_\Omega \left(D_\xi \Phi(\nabla u)-D_\xi \Phi(\nabla v)\right)\cdot\nabla\omega^k_{\varepsilon,h}\,dx\,dt=0. \tag{3.11}$$

By the same argument in the proof of Corollary 1.4, we can have

$$\frac{1}{2}\int_\Omega (u-v)(t)\omega^k(t)\,dx+\int_0^t\int_{\Omega\cap\{|u-v|<k\}}\left(D_\xi \Phi(\nabla u)-D_\xi \Phi(\nabla v)\right)\cdot\nabla(u-v)\,dx\,d\tau=0.$$

Sending $k\to\infty$, we conclude that

$$\frac{1}{2}\int_{\Omega}(u-v)^2(t)\,dx+\int_0^t\int_{\Omega}\big(D_\xi\Phi(\nabla u)-D_\xi\Phi(\nabla v)\big)\cdot\nabla(u-v)\,dx\,d\tau=0,$$

which implies $u=v$ a.e. in Ω_T. Therefore we obtain the uniqueness of weak solutions.

Next we prove the existence of weak solutions. Let n be a positive integer. Denote $h=T/n$. We construct an approximation solution sequence $\{u_h\}$ for problem (1.1). Consider the following elliptic problems

$$\begin{cases}\dfrac{u_k-u_{k-1}}{h}-\operatorname{div}\big[D_\xi\Phi(\nabla u_k)\big]=0 & \text{in } \Omega,\\ u_k=0 & \text{on } \partial\Omega,\end{cases}\tag{3.12}$$

for $k=1,2,\dots,n$. When $k=1$, it follows from Theorem 3.3 that there is a unique $u_1\in V$ satisfying (3.12). Following the same procedures, we find weak solutions $u_k\in V$ of (3.12) for $k=2,\dots,n$. It follows that, for every $\varphi\in C_0^1(\Omega)$,

$$\int_{\Omega}\frac{u_k-u_{k-1}}{h}\varphi\,dx+\int_{\Omega}D_\xi\Phi(\nabla u_k)\cdot\nabla\varphi\,dx=0,\tag{3.13}$$

and

$$\int_{\Omega}\frac{u_k-u_{k-1}}{h}u_k\,dx+\int_{\Omega}D_\xi\Phi(\nabla u_k)\cdot\nabla u_k\,dx=0.\tag{3.14}$$

Now for every $h=T/n$, we define

$$u_h(x,t)=\begin{cases}u_0(x), & t=0,\\ u_1(x), & 0<t\leqslant h,\\ \cdots, & \cdots,\\ u_j(x), & (j-1)h<t\leqslant jh,\\ \cdots, & \cdots,\\ u_n(x), & (n-1)h<t\leqslant nh=T.\end{cases}\tag{3.15}$$

By Cauchy's inequality, it follows from (3.14) that

$$\frac{1}{2}\int_{\Omega}u_k^2\,dx+h\int_{\Omega}D_\xi\Phi(\nabla u_k)\cdot\nabla u_k\,dx\leqslant\frac{1}{2}\int_{\Omega}u_{k-1}^2\,dx.\tag{3.16}$$

For each $t\in(0,T]$, there exists some $j\in\{1,\dots,n\}$ such that $t\in((j-1)h,jh]$. We add all the inequalities (3.16) for $k=1,\dots,j$ to obtain

$$\frac{1}{2}\int_{\Omega}u_k^2\,dx+h\sum_{k=1}^{j}\int_{\Omega}D_\xi\Phi(\nabla u_k)\cdot\nabla u_k\,dx\leqslant\frac{1}{2}\int_{\Omega}u_0^2\,dx.$$

By the definition of $u_h(x,t)$ we have

$$\frac{1}{2}\int_{\Omega} u_h^2(x,t)\,dx + \int_0^{jh}\int_{\Omega} D_\xi \Phi(\nabla u_h)\cdot \nabla u_h\,dx\,d\tau \leqslant \frac{1}{2}\int_{\Omega} u_0^2\,dx,$$

or

$$\frac{1}{2}\int_{\Omega} u_h^2(x,t)\,dx + \int_0^{t}\int_{\Omega} D_\xi \Phi(\nabla u_h)\cdot \nabla u_h\,dx\,d\tau \leqslant \frac{1}{2}\int_{\Omega} u_0^2\,dx.$$

Therefore, after taking the supremum over $[0,T]$, we obtain

$$\sup_{0\leqslant t\leqslant T}\int_{\Omega} u_h^2(x,t)\,dx + \int_0^{T}\int_{\Omega} D_\xi \Phi(\nabla u_h)\cdot \nabla u_h\,dx\,dt \leqslant B_0,$$

where $B_0 = 2\int_{\Omega} u_0^2\,dx$. Recalling (2.2), we have

$$\int_0^{T}\int_{\Omega} \Phi(\nabla u_h)\,dx\,dt \leqslant B_0.$$

By Lemma 2.6 we may draw a subsequence (we also denote it as the original sequence for simplicity) such that

$$u_h \rightharpoonup u, \quad \text{weakly-}* \text{ in } L^\infty\big(0,T;L^2(\Omega)\big),$$
$$u_h \rightharpoonup u, \quad \text{weakly in } L^1\big(0,T;W_0^{1,1}(\Omega)\big),$$

which follows that

$$\sup_{0\leqslant t\leqslant T}\int_{\Omega} u^2(x,t)\,dx \leqslant B_0,$$
$$\int_0^{T}\int_{\Omega} \Phi(\nabla u)\,dx\,dt \leqslant B_0.$$

(See [20, Chapter 2] or [13, Chapter 4].)

Denote

$$\zeta_h = D_\xi \Phi(\nabla u_h).$$

It follows from (2.5) that

$$\int_0^T\int_\Omega \Psi(\zeta_h)\,dx\,dt \leqslant \int_0^T\int_\Omega D_\xi\Phi(\nabla u_h)\cdot\nabla u_h\,dx\,dt \leqslant B_0.$$

Recalling Lemma 2.2 and Lemma 2.4, we conclude from Lemma 2.6 that there exists another subsequence $\{\xi_h\}$ (we also denote it by the original sequence for simplicity) such that

$$\zeta_h \rightharpoonup \zeta, \quad \text{weakly in } L^1(\Omega_T),$$

and

$$\int_0^T\int_\Omega \Psi(\zeta)\,dx\,dt \leqslant \liminf_{h\to 0}\int_0^T\int_\Omega \Psi(\zeta_h)\,dx\,dt \leqslant B_0.$$

Recalling inequality (2.4), we have

$$|\zeta\cdot\nabla u| \leqslant \Psi(\zeta)+\Phi(\nabla u)+\Phi(-\nabla u),$$

and then conclude that $\zeta\cdot\nabla u\in L^1(\Omega_T)$.

Then we claim that the function u is a weak solution of problem (1.1).

For each $\varphi\in C^1(\bar\Omega_T)$ with $\varphi(\cdot,T)=0$ and $\varphi(\cdot,t)|_{\partial\Omega}=0$, we take $\varphi(x,kh)$ as a test function in (3.12) for every $k\in\{1,2,\dots,n\}$ to have

$$\int_\Omega \frac{u_k(x)-u_{k-1}(x)}{h}\varphi(x,kh)\,dx+\int_\Omega D_\xi\Phi(\nabla u_k)\cdot\nabla\varphi(x,kh)\,dx=0.$$

Summing up all the equalities and recalling the definition of $u_h(x,t)$ in (3.15) and $\varphi(\cdot,T)=\varphi(\cdot,nh)=0$, we have

$$-\frac{1}{h}\int_\Omega u_0(x)\varphi(x,h)\,dx+\sum_{k=1}^{n-1}\int_\Omega u_k(x)\frac{\varphi(x,kh)-\varphi(x,(k+1)h)}{h}\,dx$$
$$+\sum_{k=1}^{n}\int_\Omega D_\xi\Phi(\nabla u_k)\cdot\nabla\varphi(x,kh)\,dx=0,$$

which is

$$-\int_\Omega u_0(x)\varphi(x,h)\,dx+\int_0^T\int_\Omega -u_h(x,t)\varphi_t(x,t)\,dx\,dt+\int_0^T\int_\Omega D_\xi\Phi(\nabla u_h)\cdot\nabla\varphi\,dx\,dt$$
$$+\sum_{k=1}^{n-1}\int_{kh}^{(k+1)h}\int_\Omega u_h(x,kh)\left[\frac{\varphi(x,kh)-\varphi(x,(k+1)h)}{h}+\varphi_t(x,t)\right]dx\,dt$$
$$+\sum_{k=1}^{n}h\int_\Omega D_\xi\Phi(\nabla u_k)\cdot\left[\nabla\varphi(x,kh)-\frac{1}{h}\int_{kh}^{(k+1)h}\nabla\varphi(x,t)\,dt\right]dx=0.$$

Passing to limits as $h \to 0$, we have

$$-\int_{\Omega} u_0(x)\varphi(x,0)\,dx + \int_0^T \int_{\Omega} [-u\varphi_t + \zeta \cdot \nabla\varphi]\,dx\,dt = 0. \tag{3.17}$$

Choosing $\varphi \in C_0^\infty(\Omega_T)$, we have

$$\int_0^T \int_{\Omega} u\varphi_t\,dx\,dt = \int_0^T \int_{\Omega} \zeta \cdot \nabla\varphi\,dx\,dt.$$

Since $\zeta \in L^1(\Omega_T)$, we conclude that $u_t \in L(0,T;W^{-1,1}(\Omega))$, which implies that $u \in C([0,T];H^{-s}(\Omega))$ where s is a sufficiently large positive number. Here $H^{-s}(\Omega)$ is the dual space of $H_0^s(\Omega) = W_0^{s,2}(\Omega)$.

Then we show

$$\zeta = D_\xi \Phi(\nabla u) \quad \text{a.e. in } \Omega_T.$$

Denote

$$Av = D_\xi \Phi(\nabla v)$$

for $v \in L(\Omega_T)$ with $\Phi(\nabla v) \in L(\Omega_T)$.

Summing up the inequalities (3.16), we get

$$\frac{1}{2}\int_{\Omega} u_h^2(T)\,dx + \int_0^T \int_{\Omega} A(u_h) \cdot \nabla u_h\,dx\,dt \leqslant \frac{1}{2}\int_{\Omega} u_0^2\,dx. \tag{3.18}$$

As $\Phi(\xi)$ is a convex function, we have

$$\int_0^T \int_{\Omega} (Au_h - Av) \cdot (\nabla u_h - \nabla v)\,dx\,dt \geqslant 0, \tag{3.19}$$

and obtain from (3.18) that

$$\frac{1}{2}\int_{\Omega} u_h^2(T)\,dx + \int_0^T \int_{\Omega} A(u_h) \cdot \nabla v\,dx\,dt + \int_0^T \int_{\Omega} A(v) \cdot \nabla u_h\,dx - \int_0^T \int_{\Omega} A(v) \cdot \nabla v\,dx\,dt$$
$$\leqslant \frac{1}{2}\int_{\Omega} u_0^2\,dx.$$

Passing to limits as $h \to 0$ in the above inequality and noting that

$$\int_\Omega u^2(T)\,dx \leqslant \liminf_{h\to 0} \int_\Omega u_h^2(T)\,dx,$$

we obtain

$$\frac{1}{2}\int_\Omega u^2(T)\,dx + \int_0^T\int_\Omega \zeta\cdot\nabla v\,dx\,dt + \int_0^T\int_\Omega Av\cdot\nabla u\,dx\,dt - \int_0^T\int_\Omega Av\cdot\nabla v\,dx\,dt$$
$$\leqslant \frac{1}{2}\int_\Omega u_0^2\,dx. \tag{3.20}$$

By an approximation argument, we may choose the test function $\varphi = u$ in (3.17) to have

$$\frac{1}{2}\int_\Omega u^2(T)\,dx + \int_0^T\int_\Omega \zeta\cdot\nabla u\,dx\,dt = \frac{1}{2}\int_\Omega u_0^2\,dx. \tag{3.21}$$

Combining (3.20) with (3.21), we get

$$\int_0^T\int_\Omega (\zeta - Av)\cdot(\nabla v - \nabla u)\,dx\,dt \leqslant 0. \tag{3.22}$$

Set $v = \lambda u$, $\lambda \in (0,1)$. Then

$$\int_0^T\int_\Omega \big(\zeta - D_\xi\Phi(\lambda\nabla u)\big)\cdot\nabla u\,dx\,dt \geqslant 0,$$

that is

$$\int_0^T\int_\Omega D_\xi\Phi(\lambda\nabla u)\cdot\nabla u\,dx\,dt \leqslant \int_0^T\int_\Omega \zeta\cdot\nabla u\,dx\,dt.$$

Passing to limits as $\lambda \to 1$, we conclude that

$$\int_0^T\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla u\,dx\,dt \leqslant \int_0^T\int_\Omega \zeta\cdot\nabla u\,dx\,dt < +\infty.$$

Next we choose $v = \lambda u + (1-\lambda)w$ for any $\lambda \in (0,1)$, $w \in C^1(\bar{\Omega}_T)$ in inequality (3.22) to have

$$\int_0^T \int_\Omega \big(\zeta - D_\xi \Phi\big(\lambda \nabla u + (1-\lambda)\nabla w\big)\big) \cdot (\nabla w - \nabla u)\, dx\, dt \leqslant 0.$$

It is easy to check that we can replace ∇w with any $\psi \in (L^\infty(\Omega_T))^N$. Passing to limits as $\lambda \to 1$ and using Lebesgue's dominated convergence theorem, we obtain

$$\int_0^T \int_\Omega (\zeta - Au) \cdot (\psi - \nabla u)\, dx\, dt \leqslant 0.$$

By a scaling argument again, we have

$$\int_0^T \int_\Omega (\zeta - Au) \cdot \psi\, dx\, dt = 0,$$

for every $\psi \in (L^\infty(\Omega_T))^N$. It follows that $\zeta = Au$ a.e. in Ω_T.

For every $h > 0$, we denote $v_h(x,t) = u(x, t+h)$. It follows from the uniqueness of weak solutions that v_h is a weak solution for the following problem

$$\begin{cases} (v_h)_t - \operatorname{div}\big(D_\xi \Phi(\nabla v_h)\big) = 0 & \text{in } \Omega_T, \\ v_h = 0 & \text{on } \Sigma, \\ v_h(x,0) = u(x,h) & \text{in } \Omega. \end{cases} \tag{3.23}$$

Then $w_h(x,t) = v_h(x,t) - u(x,t)$ satisfies

$$\begin{cases} (w_h)_t - \operatorname{div}\big[D_\xi \Phi(\nabla v_h) - D_\xi \Phi(\nabla u)\big] = 0 & \text{in } \Omega_T, \\ w_h = 0 & \text{on } \Sigma, \\ w_h(x,0) = u(x,h) - u_0(x) & \text{in } \Omega. \end{cases} \tag{3.24}$$

For each $t_0 \in [0,T]$, we choose test function w_h for Eq. (3.24) over $[0,t_0]$ to have

$$\begin{aligned} &\frac{1}{2}\int_\Omega w_h^2(x,t_0)\, dx + \int_0^{t_0}\int_\Omega \big[D_\xi \Phi(\nabla v_h) - D_\xi \Phi(\nabla u)\big] \cdot (\nabla v_h - \nabla u)\, dx\, d\tau \\ &\quad \leqslant \frac{1}{2}\int_\Omega w_h^2(x,0)\, dx, \end{aligned}$$

which implies that

$$\int_\Omega \big|u(x,t_0+h) - u(x,t_0)\big|^2\, dx \leqslant \int_\Omega \big|u(x,h) - u_0(x)\big|^2\, dx.$$

So, in order to prove that $u \in C([0,T], L^2(\Omega))$, we only need to prove

$$\limsup_{h\to 0^+}\int_\Omega \left|u(x,h)-u_0(x)\right|^2\,dx=0. \tag{3.25}$$

Suppose that (3.25) is not true. Then there exist a positive number δ and a sequence $\{h_i\}$ with $h_i\to 0$ as $i\to\infty$ such that

$$\lim_{h_i\to 0^+}\int_\Omega \left|u(x,h_i)-u_0(x)\right|^2\,dx\geqslant \delta. \tag{3.26}$$

As

$$\int_\Omega \left|u(x,h_i)\right|^2\,dx\leqslant \int_\Omega \left|u_0(x)\right|^2\,dx, \tag{3.27}$$

we have from (3.26) that

$$\liminf_{h_i\to 0^+}\left(\int_\Omega \left|u_0(x)\right|^2\,dx-\int_\Omega u_0(x)u(x,h_i)\,dx\right)\geqslant \frac{\delta}{2}. \tag{3.28}$$

It follows from (3.27) that $\{u(x,h_i)\}$ is a bounded sequence in $L^2(\Omega)$. Then we may draw a subsequence (we denote it by the original sequence) such that there exists a $\tilde u_0\in L^2(\Omega)$ satisfying

$$u(x,h_i)\rightharpoonup \tilde u_0(x),\quad \text{weakly in } L^2(\Omega).$$

As we have concluded that $u\in C([0,T];H^{-s}(\Omega))$, this implies

$$u(x,h_i)\to u_0(x),\quad \text{in } H^{-s}(\Omega).$$

Therefore we must have $\tilde u_0(x)=u_0(x)$ and then

$$u(x,h_i)\rightharpoonup u_0(x),\quad \text{weakly in } L^2(\Omega).$$

So, it leads to a contradiction to (3.28). Therefore, (3.25) is true and then $u\in C([0,T];L^2(\Omega))$. Thus we complete the proof of our main theorem. □

Proof of Corollary 1.4. This can be done by an approximation argument. Indeed, we first extend solution $u(x,t)$ to the initial value $u_0(x)$ when $t<0$. We next use a technic to approximate u in the spatial directions by a C_0^∞ sequence u_ε (see [2] or [14]). As $\partial\Omega$ is Lipschitz, there exists a finite open covering $\{U_i\}_{i=1}^K$ of $\partial\Omega$, corresponding positive number λ_i and vectors p_i such that the ball $B(x+\lambda_i\varepsilon p_i,\varepsilon)\subset\Omega$ for all $x\in U_i\cap\Omega$. Choose an open set $U_0\Subset\Omega$, such that $\{U_i\}_{i=0}^K$ forms a covering of Ω. Let $\{\eta_i\}_{i=0}^K$ be a smooth partition of unity corresponding to this covering. For $x\in U_i\cap\Omega$ $(i=1,\dots,K)$, denote $u_i^\varepsilon(x,t)=u(x+\lambda_i\varepsilon p_i,t)$. Using a standard mollifier, we can mollify u_i^ε in $U_i\cap\Omega$, $i\geqslant 1$ to get u_ε^i and mollify $u(x,t)$ in U_0 to get u_ε^0 for sufficiently small ε. Next, construct the approximation

$$u_\varepsilon = \sum_{i=0}^{K} \eta_i u_\varepsilon^i,$$

then introduce the time average of $u_\varepsilon(x,t)$,

$$\phi_{\varepsilon,h}(x,t) = \frac{1}{2h}\int_{t-h}^{t+h} u_\varepsilon(x,\tau)\,d\tau = \sum_{i=0}^{K}\eta_i(x)\frac{1}{2h}\int_{t-h}^{t+h} u_\varepsilon^i(x,\tau)\,d\tau = \sum_{i=0}^{K}\eta_i u_{\varepsilon,h}^i.$$

First, assume u is bounded. As $u \in C([0,T]; L^2(\Omega)) \cap L(0,T,W_0^{1,1}(\Omega))$ in Definition 1.1, we know that $\phi_{\varepsilon,h}(x,t) \in C^1(\bar{\Omega}_T)$ with $\phi_{\varepsilon,h}(\cdot,t)|_{\partial\Omega} = 0$, and may choose it as a test function φ in (1.9) to have

$$\int_\Omega u\phi_{\varepsilon,h}\,dx\bigg|_0^t - \int_0^t\int_\Omega u[\phi_{\varepsilon,h}]_t\,dx\,dt - \int_0^t\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla\phi_{\varepsilon,h}\,dx\,dt = 0.$$

We calculate

$$\begin{aligned}
I_1 &=: \int_\Omega u\phi_{\varepsilon,h}\,dx\bigg|_0^t - \int_0^t\int_\Omega u[\phi_{\varepsilon,h}]_t\,dx\,dt \\
&= \int_\Omega u(x,t)\cdot\left(\frac{1}{2h}\int_{t-h}^{t+h} u_\varepsilon(x,\tau)\,d\tau\right)dx - \int_\Omega u_0(x)\cdot\left(\frac{1}{2h}\int_{-h}^{h} u_\varepsilon(x,\tau)\,d\tau\right)dx \\
&\quad - \frac{1}{2h}\int_0^t\int_\Omega u(x,\tau)\cdot\big(u_\varepsilon(x,\tau+h) - u_\varepsilon(x,\tau-h)\big)\,dx\,d\tau.
\end{aligned}$$

Sending $\varepsilon \to 0$, we have

$$\begin{aligned}
I_1 &\to \int_\Omega u(x,t)\cdot\left(\frac{1}{2h}\int_{t-h}^{t+h} u(x,\tau)\,d\tau\right)dx - \int_\Omega u_0(x)\cdot\left(\frac{1}{2h}\int_{-h}^{h} u(x,\tau)\,d\tau\right)dx \\
&\quad - \frac{1}{2h}\int_0^t\int_\Omega u(x,\tau)\cdot\big(u(x,\tau+h) - u(x,\tau-h)\big)\,dx\,d\tau \\
&= \int_\Omega u(x,t)\left(\frac{1}{2h}\int_{t-h}^{t+h} u(x,\tau)\,d\tau\right)dx - \int_\Omega u_0(x)\left(\frac{1}{2h}\int_{-h}^{h} u(x,\tau)\,d\tau\right)dx
\end{aligned}$$

$$-\frac{1}{2h}\int_t^{t+h}\int_\Omega u(x,\tau-h)u(x,\tau)\,dx\,d\tau+\frac{1}{2h}\int_0^h\int_\Omega u(x,\tau-h)u(x,\tau)\,dx\,d\tau$$
$$=: J_1+J_2+J_3+J_4.$$

As

$$\begin{aligned}
J_2 &= -\int_\Omega u_0(x)\left(\frac{1}{2h}\int_{-h}^{h} u(x,\tau)\,d\tau\right)dx\\
&= -\frac{1}{2h}\int_\Omega u_0(x)\left(\int_0^h u(x,\tau)\,d\tau\right)dx-\frac{1}{2h}\int_\Omega u_0(x)\left(\int_{-h}^0 u(x,\tau)\,d\tau\right)dx\\
&= -\frac{1}{2h}\int_\Omega u_0(x)\left(\int_0^h u(x,\tau)\,d\tau\right)dx-\frac{1}{2}\int_\Omega \left|u_0(x)\right|^2dx
\end{aligned}$$

and

$$\begin{aligned}
J_4 &= \frac{1}{2h}\int_0^h\int_\Omega u(x,\tau-h)u(x,\tau)\,dx\,d\tau\\
&= \frac{1}{2h}\int_\Omega u_0(x)\left(\int_0^h u(x,\tau)\,d\tau\right)dx,
\end{aligned}$$

we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} I_1 &= \int_\Omega u(x,t)\left(\frac{1}{2h}\int_{t-h}^{t+h} u(x,\tau)\,d\tau\right)dx-\frac{1}{2h}\int_t^{t+h}\int_\Omega u(x,\tau-h)u(x,\tau)\,dx\,d\tau\\
&\quad -\frac{1}{2}\int_\Omega \left|u_0(x)\right|^2dx.
\end{aligned}$$

Next we send $h\to 0$ to have, for a.e. $t\in(0,T]$,

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} I_1=\frac{1}{2}\int_\Omega \left|u(x,t)\right|^2dx-\frac{1}{2}\int_\Omega \left|u_0(x)\right|^2dx.$$

Note

$$I_2=-\int_0^t\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla\phi_{\varepsilon,h}\,dx\,d\tau=\sum_{i=0}^K-\int_0^t\int_\Omega D_\xi\Phi(\nabla u)\cdot\left(u^i_{\varepsilon,h}\nabla\eta_i+\eta_i\nabla u^i_{\varepsilon,h}\right)dx\,d\tau.$$

Choosing $\xi = \nabla u^0_{\varepsilon,h}$, $\zeta = \nabla u$ and $\eta = D_\xi \Phi(\nabla u)$ in the inequalities in (2.4) and (2.5) and recalling that Φ is a convex function, we have in U_0,

$$\begin{aligned}\left|\eta_0 D_\xi \Phi(\nabla u)\cdot \nabla u^0_{\varepsilon,h}\right| &\leqslant \Phi\big(\nabla u^0_{\varepsilon,h}\big) + \Phi\big(-\nabla u^0_{\varepsilon,h}\big) + \Psi\big(D_\xi \Phi(\nabla u)\big)\\ &\leqslant \big[\Phi(\nabla u)\big]_{\varepsilon,h} + \big[\Phi(-\nabla u)\big]_{\varepsilon,h} + D_\xi \Phi(\nabla u)\cdot \nabla u\\ &\leqslant C\big[\Phi(\nabla u)\big]_{\varepsilon,h} + D_\xi \Phi(\nabla u)\cdot \nabla u.\end{aligned}$$

When $\varepsilon \to 0$ and $h \to 0$, in view of Lemma 2.5 and boundedness of u, this justifies

$$-\int_0^t \int_\Omega D_\xi \Phi(\nabla u)\cdot \eta_0 \nabla u^0_{\varepsilon,h}\,dx\,d\tau \to -\int_0^t \int_\Omega D_\xi \Phi(\nabla u)\cdot \eta_0 \nabla u\,dx\,d\tau,$$

and

$$-\int_0^t \int_\Omega D_\xi \Phi(\nabla u)\cdot u^0_{\varepsilon,h} \nabla \eta_0\,dx\,d\tau \to -\int_0^t \int_\Omega D_\xi \Phi(\nabla u)\cdot u \nabla \eta_0\,dx\,d\tau.$$

For $i = 1, \ldots, K$, the same conclusion holds. Therefore,

$$I_2 \to -\sum_{i=0}^K \int_0^t \int_\Omega D_\xi \Phi(\nabla u)\cdot (u\nabla \eta_i + \eta_i \nabla u)\,dx\,d\tau = -\int_0^t \int_\Omega D_\xi \Phi(\nabla u)\cdot \nabla u\,dx\,d\tau,$$

as $\varepsilon \to 0$ and $h \to 0$. Thus we conclude that (1.10) is true for a.e. $t \in [0, T]$.

If u is not bounded, we consider $u_k = \frac{1}{2}(|u+k| - |u-k|) = u\chi_{\{|u|\leqslant k\}} - k\chi_{\{u<-k\}} + k\chi_{\{u>k\}}$, then $u_k \in C([0,T]; L^2(\Omega)) \cap L(0,T, W_0^{1,1}(\Omega)) \cap L^\infty(\Omega_T)$. Following the above arguments, and letting $k \to \infty$, we conclude that equality (1.10) is true for a.e. $t \in [0,T]$. Since $u \in C([0,T]; L^2(\Omega))$, (1.10) holds for every $t \in [0,T]$. This finishes the proof of Corollary 1.4. □

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